MATHEMATICAL GROUND FOR THE ASYMPTOTIC NATURE OF THE VELOCITY OF A BODY MOVING IN A VISCOUS FLUID

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ABSTRACT

Stokes' law provides us with an equation for the terminal velocity of a ball but does not explicitly account for the time-dependent instantaneous velocity of the ball settling under the influence of gravity and other fluid-driven forces at any given time. The aim of this paper is to prove that a body never truly attains terminal velocity by exploiting Stokes' law, Archimedes' law, and Newton's second law. The results, as will be provided, proves the inherently asymptotic nature of velocity in such systems, while simultaneously manufacturing the instantaneous velocity expression which through integration with respect to the time variable gives us the displacement expression. To rephrase, the velocity of the sphere never accomplishes terminal velocity despite its continuous acceleration, but rather the relation is asymptotic. The proposed formulation therefore offers a straightforward and broadly applicable tool for analyzing motion in viscous environments and can be applied to problems such as underwater vehicle trajectory prediction, submarine design, and aerodynamic motion in fluids where precise velocity modeling is required.

1. INTRODUCTION

This paper aims to provide a vivid mathematical ground for the assertion that a body never truly attains terminal velocity. The equations provided by Classical Fluid Mechanics give a proper mathematical definition for terminal velocity, which will come in handy in proving the assertion as aimed by this particular paper, while ultimately also producing the equations for the precise instantaneous velocity and displacement of the body moving in the viscous medium through the process of integration.

2. THEORY

Consider a solid sphere having 'ρ' density and 'r' radius is moving downwards with a velocity 'v' in a fluid having 'σ' density and 'η' coefficient of viscosity. The instantaneous drag force or Stokes' drag exerted by the fluid on the solid sphere, according to Stokes' law, is given by:

$$F = 6\pi r\eta v$$

There will be a number of forces acting upon the sphere on different directions- upward and downward- due to the imbalance of which the sphere will accelerate towards one direction. However, after the attainment of terminal velocity, the forces acting on the opposite direction neutralize each other, due to which the sphere moves with a steady velocity.

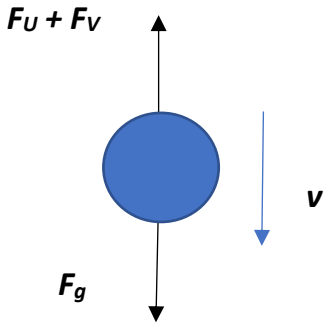

Fig. 1: Free-body diagram of a spherical ball moving through a viscous medium

The upward forces that will be acting upon the sphere are viscous drag or Stokes' drag ($\boldsymbol{F_V}$) and up-thrust ($\boldsymbol{F_U}$), whereas there will be only one downward force: weight of the sphere ($\boldsymbol{F_g}$). As we know, after the attainment of terminal velocity, the upward and downward forces will in an equilibrium, finally stabilizing the velocity of the sphere as terminal velocity (say).

Mathematically,

$$F_V + F_U = F_g$$

From Stokes' law and Archimedes' law,

$$6\pi r\eta v_t + \frac{4}{3}\pi r^3\sigma g = \frac{4}{3}\pi r^3\rho g$$

$$or, \frac{4}{3}\pi r^3(\rho - \sigma)g = 6\pi r\eta v_t$$

At any instant of time when the sphere was moving with any instantaneous velocity 'v' before the attainment of terminal velocity, the net downward force was:

$$F_a = F_g - (F_V + F_U)$$

$$or, F_a = \frac{4}{3}\pi r^3\rho g - 6\pi r\eta v - \frac{4}{3}\pi r^3\sigma g$$

$$or, F_a = \frac{4}{3}\pi r^3(\rho - \sigma)g - 6\pi r\eta v$$

From above,

$$F_a = 6\pi r\eta v_t - 6\pi r\eta v$$

$$or, F_a = 6\pi r\eta(v_t - v)$$

This equation gives the instantaneous magnitude of net downward force acting upon the sphere before attaining terminal velocity.

From Newton's second law of motion,

$$F_a = \frac{4}{3}\pi r^3\rho\frac{dv}{dt}$$

Where, a = instantaneous acceleration of the sphere

Now,

$$\frac{4}{3}\pi r^3\rho\frac{dv}{dt} = 6\pi r\eta(v_t - v)$$

$$or, \frac{dv}{dt} = \frac{9\eta(v_t - v)}{2r^2\rho}$$

$$or, \int\frac{dv}{(v_t - v)} = \int\frac{9\eta dt}{2r^2\rho}$$

$$or, -\ln(v_t - v) + c = \frac{9\eta t}{2r^2\rho}$$

Consider when ball was, initially, released into the viscous medium from its surface (at t=0), the velocity was zero (v=0). So, plugging in these values, we get:

$$-\ln(v_t) + c = 0$$

$$So, c = \ln(v_t)$$

Now,

$$-\ln(v_t - v) + \ln(v_t) = \frac{9\eta t}{2r^2\rho}$$

$$or, \ln\frac{v_t}{v_t - v} = \frac{9\eta t}{2r^2\rho}$$

$$or, \frac{v_t}{v_t - v} = e^{\frac{9\eta t}{2r^2\rho}}$$

$$So, v = v_t(1 - e^{-\frac{9\eta t}{2r^2\rho}})$$

This equation gives the instantaneous velocity of a sphere moving in a viscous medium corresponding to time.

When $v = v_t$,

$$1 = 1 - e^{-\frac{9\eta t}{2r^2\rho}}$$

$$or, e^{-\frac{9\eta t}{2r^2\rho}} = 0$$

$$or, \frac{1}{e^{\frac{9\eta t}{2r^2\rho}}} = 0$$

This equality only holds true when time (t) approaches infinity. Hence, we can conclude that a particle moving through a viscous medium never attains its terminal velocity, but terminal velocity acts as an asymptote for the velocity of the body.

Now, integrating the velocity equation with respect to time, we get the equation for displacement (say 's'):

$$\int vdt = \int v_t(1 - e^{-\frac{9\eta t}{2r^2\rho}})dt + c$$

$$or, s = v_t\int(1 - e^{-\frac{9\eta t}{2r^2\rho}})dt + c$$

$$or, s = v_t\left(t + \frac{e^{-\frac{9\eta t}{2r^2\rho}}}{\frac{9\eta}{2r^2\rho}}\right) + c$$

$$or, s = v_t\left(t + \frac{2r^2\rho}{9\eta}e^{-\frac{9\eta t}{2r^2\rho}}\right) + c$$

When t=0, s=0. So,

$$0 = \frac{2r^2\rho v_t}{9\eta} + c$$

$$So, c = -\frac{2r^2\rho v_t}{9\eta}$$

Now,

$$s = v_t\left(t + \frac{2r^2\rho}{9\eta}e^{-\frac{9\eta t}{2r^2\rho}}\right) - \frac{2r^2\rho v_t}{9\eta}$$

$$or, s = v_t\left(t + \frac{2r^2\rho}{9\eta}e^{-\frac{9\eta t}{2r^2\rho}} - \frac{2r^2\rho}{9\eta}\right)$$

$$So, s = \frac{2r^2\rho}{9\eta}.v_t\left(e^{-\frac{9\eta t}{2r^2\rho}} - 1\right) + v_t t$$

This is the required equation for displacement of the sphere.

3. DISCUSSION AND CONCLUSION

These derived equations can prove effective for submarines in order to calculate their trajectories inside a water body. Furthermore, they can be used to gain information about the quantitative aerodynamics of bodies, as air is a fluid.

Hence, the proposition that a body, spherical for the purpose of this paper, never truly attains terminal velocity is proven.

CONFLICT OF INTEREST

The author of this paper hereby discloses that there is no conflict of interest in any of his part.

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